THE MAYER-BOLZA PROBLEM FOR MULTIPLE INTEGRALS AND THE OPTIMIZATION OF THE PERFORMANCE OF SYSTEMS WITH DISTRIBUTED PARAMETERS

(ZADACHA MAIERA-BOL'TSA DLIA KRATNYKH INTEGRALOV I OPTIMIZATSIIA POVEDENIIA SISTEM S RASPREDELENNYMI PARAMETRAMI)

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K. A. LUR'E
(Leningrad)

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The problem of optimizing systems with distributed parameters has been investigated from various points of view in papers by a number of authors, for instance [1-4]. In [1] Bellman's dynamic programming method was applied to the problem; papers [2-4] contain, together with a general statement of the problem, an analog of the maximum principle proposed by L.S. Pontriagin, and moreover, in the latter set of papers (except [2]) from the very beginning the problem is formulated in terms of integral relations*.

The paper here offered has the purpose of obtaining necessary optimality conditions by the methods of classical calculus of variations. The optimal problem is formulated as a Mayer-Bolza problem for multiple integrals with connections given both by partial differential equations and also by ordinary differential equations. The necessary stationary value conditions and the necessary Weierstrass condition are obtained; from this latter condition an analog of the maximum principle is derived. For the sake of simplicity the presentation is carried out for two independent variables.

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* After the paper was submitted to the editor, the author became acquainted with the just-published paper of A.I. Egorov (*PMM Vol. 27, No.4, 1963*) containing optimality conditions for processes described by systems of quasilinear hyperbolic equations. *(Note in proof)*.

For optimal problems described by ordinary differential equations, analogous constructions were carried out in the papers of Troitskii [5,6], Berkovitz [7] and Kalman [8].

The optimization of systems with distributed parameters is investigated below within the framework of the study of solutions of correspond ing canonic systems. For the completion of a logical scheme peculiar to the variational approach, it is necessary to study another aspect of the problem, namely, a series of questions connected with Bellman's principle of optimality and with the Hamilton-Jacobi equation. Such investigations for optimal problems with ordinary differential equations are contained in the paper by Dreyfus [9] and also in the papers of Berkovitz [7] and Kalman [8] already mentioned.

1. Statement of the problem. A doubly-connected region S in the xy-plane with piecewise-smooth boundary curves Σ_1 and Σ_2 is given (Figure). In the closed region S is given a system of first order partial differential equations

$$\Xi_i \equiv \frac{\partial z^i}{\partial x} - X_i(z, \zeta, u; x, y) = 0, \qquad \mathrm{H}_i \equiv \frac{\partial z^i}{\partial y} - Y_i(z, \zeta, u; x, y) = 0$$

$$\Phi_i \equiv \frac{\partial X_i}{\partial y} - \frac{\partial Y_i}{\partial x} = 0 \tag{1.1}$$

This system consists of the components of the vector function $z = (z^1, \ldots, z^n)$, and also of the vector functions $\zeta = (\zeta^1, \ldots, \zeta^\nu)$ and $u = (u^i, \ldots, u^p)$. The functions z together with ζ give a characteristic system, while the functions u play the role of "extensional controls". The totality of functions z and ζ will be called the state of the system

Any system of partial differential equations can be reduced to the form (1.1) [10, p.324] (with an increase, if necessary, in the number of dependent variables). For example, the Helmholtz equation

$$z_{xx}^1 + z_{yy}^1 + uz^1 = 0$$

is equivalent to the system

$$z_x^1 = z^2, \quad z_y^1 = z^3, \quad z_x^2 = -\zeta^2 - uz^1, \quad z_y^2 = \zeta^1, \quad z_x^3 = \zeta^1, \quad z_y^3 = \zeta^2$$

$$z_y^2 - z_x^3 = 0, \quad \zeta_y^2 + (uz^1)_y + \zeta_x^1 = 0, \quad \zeta_y^1 - \zeta_x^2 = 0$$

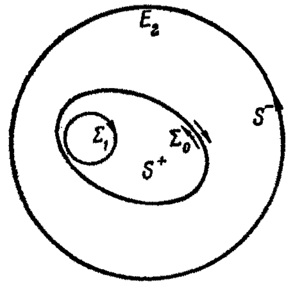
It is not difficult to see that the role of functions ζ consists, essentially, in giving the total order of the system.

With equation (1.1) are associated $r \leqslant p$ constraints imposed on the extensional controls; of these the first r_1 have the form of finite equalities

$$G_k(u; x, y) = 0 \qquad (k = 1, \ldots, r_1) \tag{1.2}$$

and the remaining $r - r_1$ are given by finite inequalities

$$G_k(u; x, y) \geqslant 0 \qquad (k = r_1 + 1, \ldots, r \leqslant p) \qquad (1.3)$$



Let us suppose that the values of the first $n_1 \leqslant n$ functions z^i are given on the curve Σ_1 which is assumed to be known. Thus

$$z^i|_{\Sigma_1} = z_1^i(t) \quad (i = 1, \ldots, n_1) \qquad (1.4)$$

The number n_1 is determined by the conditions of each actual problem.

The closed curve Σ_2 is not taken as known *a priori*; it is assumed only that on this curve are observed $n_2 \leqslant n$ first-order differential equations of the form*

$$\Theta_{i_k} \equiv \frac{dz^{i_k}}{dt} - T_{i_k}(z, v; t) = 0 \qquad (i_k = i_1, \ldots, i_{n_2}) \qquad (1.5)$$

In these equations occur, among others, the functions

$$v^\varkappa = v^\varkappa(t) \quad (\varkappa = 1, \ldots \pi)$$

called the boundary controls. The values of z^{i_k} when $t = 0$ are assumed to be known. Between the functions v^κ are established, similar to (1.2) and (1.3), relations expressed by finite equalities

$$g_k(v; t) = 0 \qquad (k = 1, \ldots, \rho_1) \qquad (1.6)$$

and inequalities

$$g_k(v; t) \geqslant 0 \qquad (k = \rho_1 + 1, \ldots, \rho \leqslant \pi) \qquad (1.7)$$

The total number of these relations equals $\rho \leqslant \pi$.

The extensional controls u^k will be taken to be piecewise continuous functions of coordinates x and y; the possible discontinuities of these functions are arranged, by assumption, along some isolated closed smooth lines Σ_0.

The superscripts plus and minus associate, respectively, the regions lying to the left and to the right of the lines of discontinuity. "Left" and "right" sides are determined in the usual fashion by the positive

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* The number n_2, similarly to n_1, is determined by the actual given problem.

direction of traverse along Σ_0 around the region enclosed by this curve. For example, in the Figure the region on the left of the discontinuity will be that which borders on Σ_0 from within.

The functions z^i will be taken to be continuous when passing through Σ_0.

In what follows, for simplicity it is assumed that in region S there is one discontinuity of the extensional controls, situated on the simple closed curve Σ_0 which can be reduced to any of the boundary curves by a continuous deformation (Figure).

In like manner the boundary controls v^κ may undergo discontinuities of the first order on curve Σ_2; for the functions v^κ the superscripts plus and minus correspond to limit values of the functions before and after the discontinuity.

For simplicity we shall consider that we have only one such point of discontinuity t_*; the functions z^i, considered on Σ_2, are assumed* to be continuous when passing through the point t_*. Generally speaking, the direction of the normal to curve Σ_2 is discontinuous at t_* since only under this condition do the derivatives dz^{ik}/dt lose continuity at t_*. The functions z^i, ζ^j and u^k are assumed to be single-valued in the closed region S.

The Mayer-Bolza problem can now be formulated in the following form. We define the functions z^i, ζ^j and the controls u^k, v^κ such that when all the conditions enumerated above are valid, the sum

$$J = \iint\limits_S F(z, \zeta, u; x, y)\, dx\, dy + \oint\limits_{\Sigma_1} f_1(z, t)\, dt + \oint\limits_{\Sigma_2} f_2(z, v, t)\, dt \tag{1.8}$$

takes the smallest possible value.

2. Necessary conditions for a stationary value of functional (1.8). First of all, in the usual manner we pass to the open region of variation of the extensional and boundary controls, introducing (real) auxiliary controls $u_* = (u_*^{r_1+1}, \ldots, u_*^r)$ and $v_*^{\rho_1+1}, \ldots, v_*^\rho)$ with the aid of the equalities

$$G_k^* = G_k(u; x, y) - (u_*^k)^2 = 0 \qquad (k = r_1 + 1, \ldots, r) \tag{2.1}$$

$$g_k^* = g_k(v; t) - (v_*^k)^2 = 0 \qquad (k = \rho_1 + 1, \ldots, \rho) \tag{2.2}$$

which replace, respectively, inequalities (1.3) and (1.7).

* We have in mind the limit values of these functions on Σ_2.

We begin to compose the necessary conditions for a stationary value by introducing the Lagrange multipliers

$$\xi_i^{\pm}(x, y), \quad \eta_i^{\pm}(x, y), \quad \varphi_i^{\pm}(x, y) \quad (i = 1, \ldots, n)$$
$$\Gamma_k^{\pm}(x, y) \quad (k = 1, \ldots, r_1), \quad \Gamma_k^{*\pm}(x, y) \quad (k = r_1 + 1, \ldots, r)$$
$$\theta_{i_k}(t) \quad (i_k = i_1, i_2, \ldots, i_{n_2})$$
$$\gamma_k(t) \quad (k = 1, \ldots, \rho_1) \quad \gamma_k^*(t) \quad (k = \rho_1 + 1, \ldots, \rho) \tag{2.3}$$

Using these multipliers we construct the functional (the products of vector functions are understood to be scalars)

$$\Pi = J + \iint_{S^+} (\xi^+\Xi^+ + \eta^+\mathrm{H}^+ + \varphi^+\Phi^+ + \Gamma^+G^+ + \Gamma^{*+}G^{*+})\, dx\, dy + \iint_{S^-} (\xi^-\Xi^- + \eta^-\mathrm{H}^- + \varphi^-\Phi^- + \Gamma^-G^- + \Gamma^{*-}G^{*-})\, dx\, dy + \oint_{\Sigma_2} (\theta\Theta + \gamma g + \gamma^* g^*)\, dt \tag{2.4}$$

The functional Π always equals J; therefore, in particular, Π and J are simultaneously stationary.

In what follows we transform the terms in (2.4) containing the factors φ by integration by parts; we obtain*

$$\iint_{S^+} \varphi^+\Phi^+\, dx\, dy = \iint_{S^+} \varphi^+ \left(\frac{\partial X^+}{\partial y} - \frac{\partial Y^+}{\partial x}\right) dx\, dy = -\left(\oint_{\Sigma_0} + \oint_{\Sigma_1}\right) \varphi^+ (X^+dx + Y^+dy) - \iint_{S^+} \left(X^+ \frac{\partial \varphi^+}{\partial y} - Y^+ \frac{\partial \varphi^+}{\partial x}\right) dx\, dy = -[\varphi^+z^+]_{\Sigma_0} - [\varphi^+z^+]_{\Sigma_1} + \left(\oint_{\Sigma_0} + \oint_{\Sigma_1}\right) z^+\varphi_t^+ dt - \iint_{S^+} \left(X^+ \frac{\partial \varphi^+}{\partial y} - Y^+ \frac{\partial \varphi^+}{\partial x}\right) dxdy. \tag{2.5}$$

Let us denote by L, l_1, l_0 and l_2 the Lagrange functions

$$L = F + \xi\Xi + \eta\mathrm{H} + Y\varphi_x - X\varphi_y + \Gamma G + \Gamma^* G^*$$
$$l_1 = f_1 + \varphi^+{}_t z^+, \quad l_0 = \varphi_t z, \quad l_2 = f_2 + \theta\Theta + \gamma g + \gamma^* g^* + \varphi_t^- z^-$$

The first variation of functional Π consists of integrals over the regions S^+ and S^-, of integrals on the curves Σ_1, Σ_0 and Σ_2, and of

* By $[f]_\Sigma$ we denote the increase in function f for a single traverse around the closed curve Σ.

terms outside the integrals.

That part of the total expression for the first variation which is represented by double integrals, has the form*

$$\iint\limits_{S+}\left(\frac{\partial L^+}{\partial z^{i+}}-\frac{\partial \xi_i^+}{\partial x}-\frac{\partial \eta_i^+}{\partial y}\right)\delta z^{i+}\,dx\,dy+\iint\limits_{S-}\left(\frac{\partial L^-}{\partial z^{i-}}-\frac{\partial \xi_i^-}{\partial x}-\frac{\partial \eta_i^-}{\partial y}\right)\delta z^{i-}\,dx\,dy+$$
$$+\iint\limits_{S+}\frac{\partial L^+}{\partial \zeta^{j+}}\delta\zeta^{j+}\,dx\,dy+\iint\limits_{S-}\frac{\partial L^-}{\partial \zeta^{j-}}\delta\zeta^{j-}\,dx\,dy+\iint\limits_{S+}\frac{\partial L^+}{\partial u^{k+}}\delta u^{k+}\,dx\,dy+$$
$$+\iint\limits_{S-}\frac{\partial L^-}{\partial u^{k-}}\delta u^{k-}\,dx\,dy+\iint\limits_{S+}\frac{\partial L^+}{\partial u_*^{k+}}\delta u_*^{k+}\,dx\,dy+\iint\limits_{S-}\frac{\partial L^-}{\partial u_*^{k-}}\delta u_*^{k-}\,dx\,dy \qquad (2.6)$$

For obvious reasons the Lagrange multipliers do not vary.

Let us consider the line integral

$$\oint\limits_{\Sigma} f\,dt$$

where f is a limit value on the smooth curve Σ of a function which is continuous together with its first derivative and is given in the region adjoining the curve. For the variation of such integrals we should follow the rule

$$\delta\oint f\,dt=\oint\delta f\,dt+\oint\left(\frac{f}{\rho}+\frac{\partial f}{\partial n}\right)\delta n\,dt \qquad (2.7)$$

where ρ is the radius of curvature of the curve and δn is the variation of the external normal (the normal in the direction outside the region, corresponding to the direction of traverse of the curve in accordance with the above-mentioned rule).

The integral on curve Σ_1 in the expression for the first variation has the form

$$\oint\limits_{\Sigma_1}\left(\frac{\partial l_1}{\partial z^{i+}}+\xi_i^+\frac{dy}{dt}-\eta_i^+\frac{dx}{dt}\right)\delta z^{i+}\,dt \qquad (2.8)$$

The integral on curve Σ_0 of discontinuity of the extensional controls is written in the form

$$\oint\limits_{\Sigma_0}\left\{\left[\left(\xi_i\frac{dy}{dt}-\eta_i\frac{dx}{dt}+\frac{\partial l_0}{\partial z^i}\right)\delta z^i\right]_-^+ +\left(L+\frac{l_0}{\rho_0}+\frac{\partial l_0}{\partial n}\right)_-^+\delta n\right\}dt \qquad (2.9)$$

* Here and in what follows we accept the usual summation condition.

To make up the integral on curve Σ_2 it is necessary to take into consideration the presence of the discontinuity of the boundary controls. In fact, this is taken into account by introducing the corner point t_* on curve Σ_2. We obtain

$$\oint_{\Sigma_2}\Big[\Big(\xi_i^-\frac{dy}{dt}-\eta_i^-\frac{dx}{dt}+\frac{\partial l_2}{\partial z^{i-}}-\frac{d\theta_{i_k}}{dt}\Big)\delta z^{i-}+ \\ +\frac{\partial l_2}{\partial v^\varkappa}\delta v^\varkappa+\frac{\partial l_2}{\partial v_*^{\varkappa}}\delta v_*^{\varkappa}+\Big(L^-+\frac{l_2}{\rho_2}+\frac{\partial l_2}{\partial n}\Big)\delta n\Big]dt+(\theta_{i_k}\delta z^{i_k})_+^- \qquad (2.10)$$

On the line of discontinuity Σ_0 the conditions

$$\delta f=\Delta f-\frac{\partial f}{\partial n}\delta n \qquad (2.11)$$

are valid, where Δ is the symbol for total variation of the function.

By hypothesis, the functions z^i are continuous on Σ_0, and the same is true of their total variations. The functions ζ^j and u^k, generally speaking, are discontinuous on Σ_0. Therefore integral (2.9) can be rewritten in the following form:

$$\oint_{\Sigma_0}\Big\{\Big(\xi_i\frac{dy}{dt}-\eta_i\frac{dx}{dt}+\frac{\partial l_0}{\partial z^i}\Big)_-^+\Delta z^i+\Big[L+\frac{l_0}{\rho_0}+\frac{\partial l_0}{\partial n}-\Big(\xi_i\frac{dy}{dt}-\eta_i\frac{dx}{dt}+ \\ +\frac{\partial l_0}{\partial z^i}\Big)\frac{\partial z^i}{\partial n}\Big]_-^+\delta n\Big\}dt \qquad (2.12)$$

At the point of discontinuity of the boundary controls on Σ_2 the equality $\delta z^i = \Delta z^i - (\text{grad } z^i \times \delta\mathbf{r})$; by $\delta\mathbf{r}$ we denote the variation of the radius vector at the corner point.

If we take into account the continuity of the total variations of functions z^i at the point t_*, then the term outside the integral in (2.1) is transformed to the form

$$[\theta_{i_k}(t_*)]_+^-\Delta z^{i_k}(t_*)-[\theta_{i_k}(t_*)\,\text{grad}\, z^{i_k}]_+^-\cdot\delta\mathbf{r} \qquad (2.13)$$

The first variation of functional Π is obtained by summing expressions (2.6), (2.8), (2.10) and (2.12), and also the corresponding terms outside the integrals with due regard to (2.13).

Usual arguments lead to the following stationary value conditions.

In the regions $S^\pm$

$$\frac{\partial \xi_i^{\pm}}{\partial x}+\frac{\partial \eta_i^{\pm}}{\partial y}-\frac{\partial L^{\pm}}{\partial z^{i\pm}}=0 \quad (i=1,\ldots,n), \qquad \frac{\partial L^{\pm}}{\partial \zeta^{j\pm}}=0 \quad (j=1,\ldots,\nu) \qquad (2.14)$$

$$\frac{\partial L^{\pm}}{\partial u^{k\pm}} = 0 \quad (k = 1, \ldots, p), \qquad \frac{\partial L^{\pm}}{\partial u_*^{k\pm}} \equiv -2\Gamma_k^{*\pm} u_*^{k\pm} = 0 \quad (k = r_1 + 1, \ldots, r)$$

Along boundary Σ_1

$$\frac{\partial l_1}{\partial z^{i+}} + \xi_i^+ \frac{dy}{dt} - \eta_i^+ \frac{dx}{dt} = 0, \quad [\varphi_i^+]_{\Sigma_1} = 0 \quad (i = n_1 + 1, \ldots, n) \tag{2.15}$$

Along boundary Σ_2

$$\frac{d\theta_{i_k}}{dt} - \frac{\partial l_2}{\partial z^{i-}} - \xi_i^- \frac{dy}{dt} + \eta_i^- \frac{dx}{dt} = 0, \qquad [\varphi_i^-]_{\Sigma_1} = 0 \, (i \neq i_k) \qquad (i = 1, \ldots, n) \tag{2.16}$$

$$\frac{\partial l_2}{dv^{\varkappa}} = 0 \quad (\varkappa = 1, \ldots, \pi)$$

$$\frac{\partial l_2}{\partial v_*^{\varkappa}} \equiv -2\gamma_{\varkappa}^* v_*^{\varkappa} = 0 \quad (\varkappa = \rho_1 + 1, \ldots, \rho), \qquad L^- + \frac{l_2}{\rho_2} + \frac{\partial l_2}{\partial n} = 0$$

At the point t_* of discontinuity of the boundary controls

$$\theta_{i_k}^-(t_*) - \varphi_{i_k}^-(t_*) = \theta_{i_k}^+(t_*) - \varphi_{i_k}^+(t_*) \quad \theta_{i_k}^-(t_*) \operatorname{grad} z^{i_k -}(t_*) = $$
$$= \theta_{i_k}^+(t_*) \operatorname{grad} z^{i_k +}(t_*) \qquad (i_k = i_1, \ldots, i_{n_2}) \tag{2.17}$$

Along the curve Σ_0 of discontinuity of the extensional controls

$$\left(\xi_i \frac{dy}{dt} - \eta_i \frac{dx}{dt} + \frac{\partial l_0}{\partial z^i}\right)_-^+ = 0 \qquad (i = 1, \ldots, n) \qquad [\varphi_i]_-^+ = 0$$
$$\left[L + \frac{l_0}{\rho_0} + \frac{\partial l_0}{\partial n} - \left(\xi_i \frac{dy}{dt} - \eta_i \frac{dx}{dt} + \frac{\partial l_0}{\partial z}\right) \frac{\partial z^i}{\partial n}\right]_-^+ = 0 \tag{2.18}$$

Using the Hadamard-Hugoniot theorem and the first of equalities (2.18), the last condition is transformed to the form

$$\left(L + \frac{l_0}{\rho_0} + \frac{\partial l_0}{\partial n}\right)_-^+ - \xi_i^+ (z_x^i)_-^+ - \eta_i (z_y^i)_-^+ - \frac{\partial l_0}{\partial z^{i+}} \left(\frac{\partial z^i}{\partial n}\right)_-^+ = 0 \tag{2.19}$$

It is clear that the original equations and boundary condition of Section 1 should be supplemented by the equations and boundary conditions (2.14) to (2.18).

3. The Hamilton form of the obtained relations. Starting from the Lagrange function

$$L = F + \xi \Xi + \eta \mathrm{H} + Y\varphi_x - X\varphi_y + \Gamma G + \Gamma^* G^* \tag{3.1}$$

we convince ourselves that the "impulses" $\partial L/\partial z_x{}^i$ and $\partial L/\partial z_y{}^i$ coincide, respectively, with the Lagrange multipliers ξ_i and η_i.

We define the Hamilton function

$$H = [z_x{}^i L_{z_x{}^i} + z_y{}^i L_{z_y{}^i} - L]_{z_x{}^i = X_i,\ z_y{}^i = Y_i} = \xi X + \eta Y - \\ - F - Y\varphi_x + X\varphi_y - \Gamma G - \Gamma^* G^* \tag{3.2}$$

The following equalities are obvious:

$$\begin{array}{llll} H_x = -L_x, & H_y = -L_y, & H_{z^i} = -L_{z^i}, & H_{\zeta^j} = -L_{\zeta^j} \\ H_{u^k} = -L_{u^k}, & H_{u_*{}^k} = -L_{u_*{}^k}, & H_{\xi_i} = X_i, & H_{\eta_i} = Y_i \end{array} \tag{3.3}$$

Using these relations we replace the first pair of equalities of (1.1) and the equalities of (2.14) by the following formulas:

$$z_x{}^i = \frac{\partial H}{\partial \xi_i}, \qquad z_y{}^i = \frac{\partial H}{\partial \eta_i}, \qquad \frac{\partial \xi_i}{\partial x} + \frac{\partial \eta_i}{\partial y} = -\frac{\partial H}{\partial z^i} \tag{3.4}$$

These equations have the form of the canonic equations of Volterra [11]. The third group of equations in (1.1) play as before the role of integrability conditions.

The last three equations of (2.14) are written, respectively, in the form

$$\frac{\partial H}{\partial \zeta^j} = 0 \quad (j = 1, \ldots, \nu), \qquad \frac{\partial H}{\partial u^k} = 0 \quad (k = 1, \ldots, p)$$
$$\frac{\partial H}{\partial u_*{}^k} \equiv 2\Gamma_k{}^* u_*{}^k = 0 \quad (k = r_1 + 1, \ldots, r) \tag{3.5}$$

Condition (2.19) is rewritten as

$$(H)_-{}^+ = z_x{}^{i-} (\xi_i)_-{}^+ + z_y{}^{i-} (\eta_i)_-{}^+ + \left(\frac{l_0}{\rho_0} + \frac{\partial l_0}{\partial n}\right)_-^+ - \frac{\partial l_0{}^+}{\partial z^{i+}} \left(\frac{\partial z^i}{\partial n}\right)_-^+ \tag{3.6}$$

Likewise, starting from the expression for the Lagrange function

$$l_2 = f_2 + \theta\Theta + \gamma g + \gamma^* g^* + \varphi_t{}^- z^- \tag{3.7}$$

we discover that the "impulses" $\partial l / \partial z_t{}^{i_k}$ coincide with the Lagrange multipliers θ_{i_k}; let us define the Hamilton function

$$h = [z_t{}^i l_{2 z_t{}^i} - l_2]_{z_t{}^{i_k} = T_{i_k}} = \theta_{i_k} T_{i_k} - f_2 - \gamma g - \gamma^* g^* - \varphi_t{}^- z^- \tag{3.8}$$

As before we pass to the relations

$$h_t = -l_{2t}, \quad h_{z^i} = -l_{2z^i} \quad h_{v^\varkappa} = -l_{2v^\varkappa}, \quad h_{v_*{}^\varkappa} = -l_{2v_*{}^\varkappa}, \quad h_{\theta i_k} = T_{i_k} \tag{3.9}$$

Keeping this in mind, we replace equations (1.5), and also those of the first equality of (2.16) in which $i = i_k$, by a system of relations of the form

$$\frac{dz^{i_k}}{dt} = \frac{\partial h}{\partial \theta_{i_k}}, \qquad \frac{d\theta_{i_k}}{dt} = -\frac{\partial h}{\partial z^{i_k}} + \xi_{i_k}^{-} \frac{dy}{dt} - \eta_{i_k}^{-} \frac{dx}{dt} \tag{3.10}$$

The remaining equations of (2.16), with the exception of the last, are rewritten in the form

$$\frac{\partial h}{\partial v^{\varkappa}} = 0 \quad (\varkappa = 1, \ldots, \pi), \qquad \frac{\partial h}{\partial v_*^{\varkappa}} \equiv 2\gamma^* v_*^{\varkappa} = 0 \quad (\varkappa = \rho_1 + 1, \ldots, \rho) \tag{3.11}$$

4. The necessary conditions of Weierstrass and Clebsch. These conditions may be derived by a single method for the classic Mayer-Bolza problem and for the problem containing controls.

Let us introduce the Weierstrass functions

$$\begin{aligned} E^{(1)} = {} & L(z, Z_x, Z_y, Z, U, U_*; \xi, \eta, \varphi_x, \varphi_y, \Gamma, \Gamma^*) - \\ & - L(z, z_x, z_y, \zeta, u, u_*; \xi, \eta, \varphi_x, \varphi_y, \Gamma, \Gamma^*) - \\ & - (Z_x^i - z_x^i) \frac{\partial L}{\partial z_x^i} - (Z_y^i - z_y^i) \frac{\partial L}{\partial z_y^i} \end{aligned} \tag{4.1}$$

$$\begin{aligned} E^{(2)} = {} & l_2(z, Z_t, V, V_*; \theta, \varphi_t, \gamma, \gamma^*) - \\ & - l_2(z, z_t, v, v_*; \theta, \varphi_t, \gamma, \gamma^*) - (Z_t^{i_k} - z_t^{i_k}) \frac{\partial l_2}{\partial z_t^{i_k}} \end{aligned} \tag{4.2}$$

In these formulas z, ζ, u and v correspond to the extremal and its boundary, and Z, Z, U and V are any admissible functions satisfying the condition of Section 1.

The necessary condition of Weierstrass for a strong relative minimum is given by the relations

$$E^{(1)} \geqslant 0, \qquad E^{(2)} \geqslant 0 \tag{4.3}$$

the proof of which is given in the Appendix.

Conditions (4.3) may be rewritten in the form of inequalities for the Hamilton functions

$$H(z, Z, U, U_*; \xi, \eta, \varphi_x, \varphi_y, \Gamma, \Gamma^*) \leqslant H(z, \zeta, u, u_*; \xi, \eta, \varphi_x, \varphi_y, \Gamma, \Gamma^*) \tag{4.4}$$

$$h(z, V, V_*; \theta, \varphi_t, \gamma, \gamma^*) \leqslant h(z, v, v_*; \theta, \varphi_t, \gamma, \gamma^*) \tag{4.5}$$

The auxiliary controls do not actually enter into these inequalities, since the terms containing them in the Hamilton functions equal zero.

The totality of formulas (3.5) and (4.4), and also (3.11) and (4.5), form the analog of L.S. Pontriagin's maximum principle for our problem.

The necessary conditions of Clebsch for a weak minimum are derived in the usual manner from the Weierstrass condition. Namely, let δz_x, δz_y, $\delta\zeta$, δu and δv be small variations and let

$$Z_x = z_x + \delta z_x, \qquad Z_y = z_y + \delta z_y, \qquad Z = \zeta + \delta\zeta \qquad (4.6)$$
$$U = u + \delta u, \qquad V = v + \delta v$$

We arrive at the following expressions for the Weierstrass functions (terms of order greater than two in smallness of variation are neglected):

$$E^{(1)} = -\frac{\partial^2 H}{\partial\zeta^j\,\partial\zeta^{j'}}\,\delta\zeta^j\,\delta\zeta^{j'} - 2\,\frac{\partial^2 H}{\partial\zeta^j\,\partial u^k}\,\delta\zeta^j\,\delta u^k - \frac{\partial^2 H}{\partial u^k\,\delta u^{k'}}\,\delta u^k\,\delta u^{k'} \qquad (4.7)$$
$$E^{(2)} = -\frac{\partial^2 h}{\partial v^\varkappa\,\partial v^{\varkappa'}}\,\delta v^\varkappa\,\delta v^{\varkappa'} \qquad (4.8)$$

By substituting these expressions into inequalities (4.3) we arrive at the necessary conditions of Clebsch.

The variations of the functions, entering into the Clebsch conditions, are related by a system of equations obtained by varying the equations of Section 1 in accordance with the derivatives of z^i, the functions ζ^j and the controls u^k and v^κ, namely

$$\delta z_x{}^i - \frac{\partial X_i}{\partial\zeta^j}\,\delta\zeta^j - \frac{\partial X_i}{\partial u^k}\,\delta u^k = 0, \quad \delta z_y{}^i - \frac{\partial Y_i}{\partial\zeta^j}\,\delta\zeta^j - \frac{\partial Y_i}{\partial u^k}\,\delta u^k = 0$$
$$\frac{\partial}{\partial y}\left(\frac{\partial X_i}{\partial\zeta^j}\,\delta\zeta^j + \frac{\partial X_i}{\partial u^k}\,\delta u^k\right) - \frac{\partial}{\partial x}\left(\frac{\partial Y_i}{\partial\zeta^j}\,\delta\zeta^j + \frac{\partial Y_i}{\partial u^k}\,\delta u^k\right) = 0 \qquad (4.9)$$
$$\frac{\partial G_k}{\partial u^k}\,\delta u^k = 0 \qquad (4.10)$$
$$\delta z_t{}^{i_k} - \frac{\partial T}{\partial v^\varkappa}\,\delta v^\varkappa = 0 \qquad (4.11)$$
$$\frac{\partial g_k}{\partial v^\varkappa}\,\delta v^\varkappa = 0 \qquad (4.12)$$

We assume that the auxiliary controls have already been introduced; the asterisks in their notation are discarded.

5. *Appendix.* **Necessary condition of Weierstrass.** We shall assume that the constraints imposed on the extensional and boundary controls have already been written in the form of equalities of type (1.2) and (1.6).

The following hypothesis is made as the basis of our reasoning: the extremal surface S with boundary curves Σ_1 and Σ_2 can be enclosed in an $(n_1 + n_2)$-parameter family of surfaces $S(b)$ along which are defined the functions

$$z^i(b;\ x,\ y)\quad (i=1,\ldots,n)\qquad \zeta^j(b;\ x,\ y)\quad (j=1,\ldots,\nu)$$
$$u^k(b,\ x,\ y)\quad (k=1,\ldots p) \tag{5.1}$$

The stated surfaces have boundary curves Σ_1 and $\Sigma_2(b)$, and moreover, along the latter are defined the functions

$$x\,(b;\ t),\quad y\,(b;\ t),\quad z_2^i\,(b;\ t),\quad \zeta_2^j\,(b;\ t),\quad u_2^k\,(b;\ t),\ v^\varkappa\,(b;\ t) \tag{5.2}$$

Both families have been defined such that equations (1.1), (1.2), (1.5) and (1.6) are satisfied, and such that when $b_1 = b_2 = \ldots = b_{n_1+n_2} = 0$, we arrive at functions relating to the original extremal and its boundaries Σ_1 and Σ_2.

In the region S of the xy-plane let us select the closed smooth curve Σ' bounding a region S' and not intersecting curve Σ_0 (see Sections 1 and 2); simultaneously, let us select on curve Σ_2 the point t' different from the corner point*.

Let us enclose curve Σ' from the outside by a nearby curve $\Sigma_e{}'$, located on the same extremal and not intersecting the first curve; the region between these curves will be denoted by $S_e{}' - S'$.

The equations of curves Σ' and $\Sigma_e{}'$ have the form

$$(\Sigma')\qquad x = x'\,(t),\qquad y = y'\,(t)$$
$$(\Sigma_e{}')\qquad x = x'\,(t) + e\cos\,(nx),\qquad y = y'\,(t) + e\cos\,(ny)$$

Here $e > 0$ is a parameter and $\cos(nx)$ and $\cos(ny)$ are direction cosines of the external normal to curve Σ'. When $e = 0$, curves Σ' and $\Sigma_e{}'$ coincide and the region $S_e{}' - S'$ vanishes.

The part of region S lying outside the curve $\Sigma_e{}'$ is denoted by $S_b - S_e{}'$.

---

* Here and in what follows we use one and the same notation for the regions (curves) and for their projections on the xy-plane.

Let us construct three families of surfaces

$$
\begin{array}{llll}
z^i(b;\,x,\,y), & \zeta^j(b;\,x,\,y), & u^k(b;\,x,\,y) & (x,\,y)\in S' \\
Z^i(b;\,x,\,y), & Z^j(b;\,x,\,y), & U^k(b;\,x,\,y) & (x,\,y)\in S'_e-S' \\
z^i(b;\,e,\,x,\,y), & \zeta^j(b,\,e;\,x,\,y), & u^k(b;\,x,\,y) & (x,\,y)\in S_b-S'_e
\end{array}
\tag{5.3}
$$

of which the first and the third satisfy equations (1.1) and (1.2), and the second the same equations with z^i replaced by Z^i, etc.

Likewise, let us set off on the boundary curve Σ_2 $(0\leqslant t\leqslant t_2)$ a line segment ε from the point t' in the positive direction and let us construct two families of curves complementing each other up to the closed boundary curve $\Sigma_e(b,\ \varepsilon)$. Along these families let us define

$$
\begin{array}{lllll}
Z_2^i(b,\,e;\,t), & Z_2^j(b,\,e;\,t), & u_2^k(b;\,t), & V^\varkappa(b;\,t) & (t'\leqslant t\leqslant t'+\varepsilon) \\
z_2^i(b,\,e,\,\varepsilon;\,t), & \zeta_2^j(b,\,e,\,\varepsilon;\,t), & u_2^k(b;\,t), & v^\varkappa(b;\,t) & \begin{pmatrix}0<t<t' \\ t'+\varepsilon<t<t_2\end{pmatrix}
\end{array}
\tag{5.4}
$$

Functions (5.4) satisfy boundary conditions (1.5) and (particularly) the initial conditions when $t = 0$ (see Section 1).

Let us introduce the following notations:

$$
\begin{array}{llll}
\delta_b z_\alpha{}^i=\dfrac{\partial z^i}{\partial b_\alpha}, & \delta_b\zeta_\alpha{}^j=\dfrac{\partial\zeta^j}{\partial b_\alpha}, & \delta_b u_\alpha{}^k=\dfrac{\partial u^k}{\partial b_\alpha}, & \delta_b v_\alpha{}^\varkappa=\dfrac{\partial v^\varkappa}{\partial b_\alpha} \\
\delta_e z^i=\dfrac{\partial z^i}{\partial e}, & \delta_e\zeta^j=\dfrac{\partial\zeta^j}{\partial e}, & \delta_\varepsilon z^i=\dfrac{\partial z^i}{\partial\varepsilon}, & \delta_\varepsilon\zeta^j=\dfrac{\partial\zeta^j}{\partial\varepsilon}
\end{array}
\tag{5.5}
$$

$$(\alpha=1,\ldots,n_1+n_2)$$

The families of functions constructed above obey the conditions:

families (5.3)

$$
\begin{gathered}
Z^i(b;\,x,\,y)\,|_{\Sigma'}=z^i(b;\,x,y)\,|_{\Sigma'} \\
Z^i(b;\,x,\,y)\,|_{\Sigma_e'}=z^i(b,\,e;\,x,\,y)\,|_{\Sigma_e'} \\
\operatorname{grad}(Z^i-z^i)\,|_{\Sigma'}=\mathbf{n}\,\delta_e z^i\,|_{\Sigma'}
\end{gathered}
\tag{5.6}
$$

families (5.4)

$$
\begin{gathered}
Z_2^i(b,\,e;\,t')=z_2^i(b,\,e,\,\varepsilon;\,t') \\
Z_2^i(b,\,e;\,t'+\varepsilon)=z_2^i(b,\,e,\,\varepsilon;\,t'+\varepsilon) \\
Z_{2t}^i(t')=z_{2t}^i(t')+\delta_\varepsilon z_2^i(t')
\end{gathered}
\tag{5.7}
$$

Let us consider the functional

$$\Pi(b, e, \varepsilon) = \iint_{s'} L[z(b;\ x, y), z_x(b; x, y), z_y(b; x, y), \zeta(b;\ x, y)\, u(b; x, y)]\, dxdy + \tag{5.8}$$
$$+ \iint_{s'_e - s'} L[Z(b;\ x, y), Z_x(b;\ x, y), Z_y(b;\ x, y), Z(b; x, y), U(b;\ x, y)]\, dxdy +$$
$$+ \iint_{s_b - s_e'} L[z(b, e; x, y), z_x(b, e; x, y), z_y(b, e; x, y), \zeta(b, e; x, y),\ u(b;\ x, y)]\, dxdy +$$
$$+ \oint_{\Sigma_1} l_1[z(b; t)]\, dt + \Big[\int_0^{t'} + \int_{t'+\varepsilon}^{t_2}\Big] l[z_2(b, e, \varepsilon; t)$$
$$z_{2t}(b, e, \varepsilon; t), v(b, t)\, t]\, dt + \int_{t'}^{t'+\varepsilon} l[Z_2, Z_{2t}, V, t]\, dt - [\varphi^+ z^+]_{\Sigma_1} [\varphi^- z^-]_{\Sigma_2}$$

differing from the original functional J in terms equal to zero.

Taking the stationary value conditions into account, we find

$$\frac{\partial \Pi}{\partial b_\alpha}\Big|_{e=\varepsilon=0} = \oint_{\Sigma_1} \sum_{i=1}^{n_1} \Big[\frac{\partial l_1}{\partial z^i} + \xi_i \frac{dy}{dt} - \eta_i \frac{dx}{dt}\Big] \delta_b z_\alpha{}^i\, dt + [\theta_{i_k} \delta_b z_\alpha^{ik}]_{\Sigma_2} -$$
$$- [\varphi_i{}^+ \delta_b z_\alpha{}^-]_{\Sigma_1} - [\varphi_i \delta_b z_\alpha{}^i]_{\Sigma_2} \tag{5.9}$$

Moreover

$$\frac{\partial \Pi}{\partial e}\Big|_{e=\varepsilon=0} = \oint_{\Sigma'} \Big[L(z, Z_x, Z_y, Z, U) - L(z, z_x, z_y, \zeta, u) -$$
$$- \Big(\frac{\partial L}{\partial z_x^i} \delta_e z^i \cos(nx) + \frac{\partial L}{\partial z_y{}^i} \delta_e z^i \cos(ny)\Big] dt + [\theta_{i_k} \delta_e z^{ik}]_{\Sigma_2} - [\varphi \delta_e z^i{}_\alpha]_{\Sigma_2} \tag{5.10}$$

$$\frac{\partial \Pi}{\partial \varepsilon}\Big|_{e=\varepsilon=0} = l_2(z_2, Z_{2t}, Z_2, U_2, V)\,|_{t'} - l_2(z_2, z_{2t}, \zeta_2, u_2, v)\,|_{t'} -$$
$$- \frac{\partial l}{\partial z_{2t}{}^i}\Big|_{t'} \delta_\varepsilon z_2{}^i(t') + [\theta_{i_k} \delta_\varepsilon z^{ik}]_{\Sigma_2} - [\varphi \delta_\varepsilon z^i{}_\alpha]_{\Sigma_2} \tag{5.11}$$

We take advantage of boundary conditions (1.4) and of the given values of $z^{ik}(0)$ on Σ_2. These conditions establish, firstly, n_1 finite equations connecting the parameters b_α, and secondly, n_2 finite equations connecting the parameters b_α, e and ε. We assume that from these equations all the b_α can be determined as functions of e and ε. Now, by making up the total differential of functional Π when $e = \varepsilon = 0$

$$d\Pi = \frac{\partial \Pi}{\partial e}\Big|_{e=\varepsilon=0} de + \frac{\partial \Pi}{\partial \varepsilon}\Big|_{e=\varepsilon=0} d\varepsilon =$$
$$= \Big[\frac{\partial \Pi}{\partial e} + \frac{\partial \Pi}{\partial b} \frac{\partial b}{\partial e}\Big]_{e=\varepsilon=0} de + \Big[\frac{\partial \Pi}{\partial \varepsilon} + \frac{\partial \Pi}{\partial b} \frac{\partial b}{\partial \varepsilon}\Big]_{e=\varepsilon=0} d\varepsilon$$

and the total differentials (equal to zero) of the left-hand sides of the above-mentioned finite equalities, we discover with the aid of (5.6), (5.7), (5.9) to (5.11) and (4.1) to (4.2), that

$$d\Pi = \left[\oint_{\Sigma'} E^{(1)} dt\right] de + E^{(2)}(t')\, d\varepsilon \qquad (5.12)$$

In order that functional Π, and together with it also J, reaches a minimum, it is necessary to satisfy the inequality

$$d\Pi \geqslant 0$$

The admissible surfaces (boundary curves) always correspond to positive values of parameter e or ε, or, what is the same, of the differentials de and $d\varepsilon$. This circumstance, together with (5.12), is equivalent to the requirements

$$\oint_{\Sigma'} E^{(1)} dt \geqslant 0, \qquad E^{(2)}(t') \geqslant 0$$

If we consider the arbitrariness of the choice of curve Σ' on the extremal and of the point t' on its boundary, then the obtained inequalities reduce to the necessary condition of Weierstrass.

Although the proof was carried out for those parts of the extremal and its boundary which do not contain the corner lines (points), it remains in force also for these lines (points), thanks to continuity arguments.

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